

MINIMAL ENERGY SOLUTIONS FOR REPULSIVE NONLINEAR SCHRÖDINGER SYSTEMS

RAINER MANDEL

ABSTRACT. In this paper we establish existence and nonexistence results concerning fully nontrivial minimal energy solutions of the nonlinear Schrödinger system

$$\begin{aligned} -\Delta u + u &= |u|^{2q-2}u + b|u|^{q-2}u|v|^q \quad \text{in } \mathbb{R}^n, \\ -\Delta v + \omega^2 v &= |v|^{2q-2}v + b|u|^q|v|^{q-2}v \quad \text{in } \mathbb{R}^n. \end{aligned}$$

We consider the repulsive case $b < 0$ and assume that the exponent q satisfies $1 < q < \frac{n}{n-2}$ in case $n \geq 3$ and $1 < q < \infty$ in case $n = 1$ or $n = 2$. For space dimensions $n \geq 2$ and arbitrary $b < 0$ we prove the existence of fully nontrivial nonnegative solutions which converge to a solution of some optimal partition problem as $b \rightarrow -\infty$. In case $n = 1$ we prove that minimal energy solutions exist provided the coupling parameter b has small absolute value whereas fully nontrivial solutions do not exist if $1 < q \leq 2$ and b has large absolute value.

1. INTRODUCTION

In this paper we are interested in fully nontrivial minimal energy solutions of the system

$$(1) \quad \begin{aligned} -\Delta u + u &= |u|^{2q-2}u + b|u|^{q-2}u|v|^q \quad \text{in } \mathbb{R}^n, \\ -\Delta v + \omega^2 v &= |v|^{2q-2}v + b|u|^q|v|^{q-2}v \quad \text{in } \mathbb{R}^n \end{aligned}$$

for parameter values $\omega \geq 1$ and $b \leq 0$. We henceforth assume that the exponent q satisfies $1 < q < \frac{n}{n-2}$ when $n \geq 3$ and $1 < q < \infty$ when $n = 1$ or $n = 2$. For applications in physics the special case $q = 2$ and $n \in \{1, 2, 3\}$ is of particular importance. For example, in photonic crystals the system (1) is used to describe the approximate shape of so-called band gap solitons which are special nontrivial solitary wave solutions $E(x, t) = e^{-i\kappa t}\phi(x)$ of the time-dependent nonlinear Schrödinger equation (or Gross-Pitaevski equation)

$$i\partial_t E = -\Delta E + V(x)E - |E|^2 E \quad \text{in } [0, \infty) \times \mathbb{R}^n.$$

For a detailed exposition on that matter we refer to [7].

During the last ten years many authors contributed to a better understanding of such nonlinear Schrödinger systems and various interesting results concerning the existence of nontrivial solutions have been proved using Ljusternik-Schnirelman theory [15], constrained minimization methods [1], [6], [11], [12], [14] or bifurcation theory [2]. In the case of a positive coupling parameter b many existence results for positive solutions of (1) have been proved by investigations of appropriate constrained minimization problems. For instance Maia, Montefusco,

Date: 15.03.2013.

2000 Mathematics Subject Classification. Primary: 35J50, 35J57.

Key words and phrases. Variational methods for elliptic systems.

Pellacci [11] proved the existence of nonnegative ground states of (1) which, by definition, are solutions of minimal energy among all nontrivial solutions. Here, the energy corresponds to the Euler functional I associated to (1) which is given by

$$I(u, v) = \frac{1}{2}(\|u\|^2 + \|v\|_\omega^2) - \frac{1}{2q}(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2b\|uv\|_q^q)$$

where $\|\cdot\|_{2q}, \|\cdot\|_q$ denote Lebesgue norms and $\|\cdot\|, \|\cdot\|_\omega$ denote Sobolev space norms that we will define in (9). Moreover the authors gave sufficient conditions and necessary conditions for ground states to be positive in both components which basically require the coupling parameter b to be positive and sufficiently large. In the special case $q = 2$ additional sufficient conditions for the existence of positive ground states have been proved in [1], [6]. Furthermore, for $q = 2$ and small positive values of b Lin, Wei [9], [10] and Sirakov [12] proved the existence of positive solutions which have minimal energy among all fully nontrivial solutions. From a technical point of view the approaches followed in [11] and [12], [9], [10] differ in the following way. In [11] ground states are obtained by minimizing the Euler functional I over the entire Nehari manifold

$$\mathcal{N}_b = \left\{ (u, v) : u, v \in H^1(\mathbb{R}^n), (u, v) \neq (0, 0), \|u\|^2 + \|v\|_\omega^2 = \|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2b\|uv\|_q^q \right\}$$

whereas the positive solutions found in [12], [9], [10] are minimizers of I over the subset \mathcal{M}_b of the Nehari manifold which is given by

$$\mathcal{M}_b = \left\{ (u, v) : u, v \in H^1(\mathbb{R}^n), u, v \neq 0, \|u\|^2 = \|u\|_{2q}^{2q} + b\|uv\|_q^q, \|v\|_\omega^2 = \|v\|_{2q}^{2q} + b\|uv\|_q^q \right\}.$$

When b is negative, however, the analysis of these constrained minimization problems does not produce any fully nontrivial solutions. Indeed, for $b < 0$ the minimizers of $I|_{\mathcal{N}_b}$ are given by the semitrivial solutions $(\pm u_0, 0)$ or $(0, \pm u_0)$ (the latter being possible only for $\omega = 1$) where u_0 is the unique positive function satisfying $-\Delta u_0 + u_0 = u_0^{2q-1}$ in \mathbb{R}^n , cf. [11], [8]. Furthermore it is known that $I|_{\mathcal{M}_b}$ does not admit minimizers in case $b < 0$, cf. Theorem 1 in [9]. Therefore the case of negative coupling parameters $b < 0$ has to be treated differently. In [12] Sirakov considered the minimization problem

$$(2) \quad \kappa_b^* := \inf_{\mathcal{M}_b^*} I \quad \text{where } \mathcal{M}_b^* = \{(u, v) \in \mathcal{M}_b : u, v \text{ are radially symmetric}\}$$

and he proved the existence of a minimizer of $I|_{\mathcal{M}_b^*}$ for parameter values $q = 2$ and $n \in \{2, 3\}$, cf. Theorem 2 (i). Let us note that the indispensable condition $n \geq 2$ is missing in the statement of that theorem.

The aim of this paper is to generalize Sirakov's result to all space dimensions and to the full range of superlinear and subcritical exponents. In Theorem 1 we first investigate the case $n \geq 2$. We show that minimizers (u_b, v_b) of the functional $I|_{\mathcal{M}_b^*}$ exist and that, at least up to a subsequence, these minimizers of $I|_{\mathcal{M}_b^*}$ converge to a function (\bar{u}, \bar{v}) with $\bar{u}\bar{v} = 0$ and

$$(3) \quad -\Delta \bar{u} + \bar{u} = \bar{u}^{2q-1} \quad \text{in } \{\bar{u} \neq 0\}, \quad -\Delta \bar{v} + \omega^2 \bar{v} = \bar{v}^{2q-1} \quad \text{in } \{\bar{v} \neq 0\}$$

that solves the optimal partition problem

$$(4) \quad \kappa_{-\infty}^* := \inf \left\{ \frac{1}{2} (\|u\|^2 + \|v\|_\omega^2) - \frac{1}{2q} (\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q}) : (u, v) \in \mathcal{M}_{-\infty}^* \right\}$$

where the set $\mathcal{M}_{-\infty}^*$ is defined by

$$(5) \quad \mathcal{M}_{-\infty}^* = \left\{ (u, v) : u, v \in H_r^1(\mathbb{R}^n), u, v \neq 0, uv \equiv 0, \|u\|^2 = \|u\|_{2q}^{2q}, \|v\|_\omega^2 = \|v\|_{2q}^{2q} \right\}.$$

Here, $H_r^1(\mathbb{R}^n)$ denotes the space of radially symmetric functions lying in $H^1(\mathbb{R}^n)$. In particular, we find that the supports of u_b, v_b separate as $b \rightarrow -\infty$. In general bounded domains $\Omega \subset \mathbb{R}^n$ these phenomena have been extensively studied in [3], [4], [5], [13] and our Theorem 1 can be considered as one kind of extension of their results.

In case $n = 1$, however, the situation turns out to be different. Since the embedding $H_r^1(\mathbb{R}^n) \rightarrow L^{2q}(\mathbb{R}^n)$ is not compact for $n = 1$ the existence of minimizers of $I|_{\mathcal{M}_b^*}$ cannot be proved the same way as in the case $n \geq 2$. Therefore we approximate the original problem (2) by the corresponding problem on intervals $B_R = (-R, R)$ for large $R > 0$. In Theorem 2 we show that for negative coupling parameters b with small absolute value the corresponding minimizers converge to a minimizer of $I|_{\mathcal{M}_b^*}$ as $R \rightarrow \infty$. For negative b with large absolute value, however, we prove in Theorem 3 that solutions of (1) do not exist at least for exponents $1 < q \leq 2$.

Let us present the main results of this paper. The first one deals with the case $n \geq 2$.

Theorem 1. *Let $n \geq 2, \omega \geq 1$. Then the following holds:*

- (i) *The value $\kappa_{-\infty}^*$ is attained at a nonnegative fully nontrivial solution of (3).*
- (ii) *For $b \leq 0$ the value κ_b^* is attained at a nonnegative fully nontrivial solution of (1).*
- (iii) *As $b \rightarrow -\infty$ we have $\kappa_b^* \rightarrow \kappa_{-\infty}^*$ and every sequence of minimizers of $I|_{\mathcal{M}_b^*}$ has a subsequence (u_b, v_b) such that $|b|^{1/q} u_b v_b \rightarrow 0$ in $L^q(\mathbb{R}^n)$ and $(u_b, v_b) \rightarrow (\bar{u}, \bar{v})$ where the latter function is a fully nontrivial solution of (3) with $\bar{u}\bar{v} = 0$.*

Since the proof of Theorem 1 makes extensive use of the fact that $H_r^1(\mathbb{R}^n)$ embeds compactly into $L^{2q}(\mathbb{R}^n)$ when $n \geq 2$ one has to resort to different methods when the space dimension is one. In Theorem 3 we show that there is a threshold value $b^*(\omega, q) \in [-\infty, 0)$ such that κ_b^* is attained whenever $0 \geq b > b^*(\omega, q)$ whereas κ_b^* is not attained for $b < b^*(\omega, q)$. Moreover we find that $b^*(\omega, q)$ has the variational characterization

$$(6) \quad b^*(\omega, q) = \inf_{\alpha > 0} \max \frac{(2 + \omega^{\frac{q+1}{q-1}})^{1-q} \|u_0\|^{-2q} \|u_0\|_{2q}^{2q} (\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q - \|u\|_{2q}^{2q} - \alpha^{2q} \|v\|_{2q}^{2q}}{2\alpha^q \|uv\|_q^q}$$

where the infimum is taken over all $u, v \in H_r^1(\mathbb{R})$ with $uv \neq 0$. As above the function u_0 appearing in (6) denotes the positive solution of $-\Delta u + u = u^{2q-1}$ in \mathbb{R}^n . Our first result dealing with the case $n = 1$ reads as follows.

Theorem 2. *Let $n = 1, \omega \geq 1$. Then the following holds:*

- (i) *We have $\kappa_{-\infty}^* = (2 + \omega^{\frac{q+1}{q-1}}) I(u_0, 0)$ and $\kappa_{-\infty}^*$ is not attained at any element of $\mathcal{M}_{-\infty}^*$.*
- (ii) *If $b < b^*(\omega, q)$ then $\kappa_b^* = \kappa_{-\infty}^*$ and κ_b^* is not attained at any element of \mathcal{M}_b^* .*

(iii) If $0 \geq b > b^*(\omega, q)$ then $\kappa_b^* < \kappa_{-\infty}^*$ and κ_b^* is attained at a nonnegative fully nontrivial solution of (1).

In view of part (iii) we may prove an explicit sufficient condition for the existence of a fully nontrivial solution of (1) by estimating the value $b^*(\omega, q)$ from above. To this end we use $(u, v) = (u_0, u_0(\omega \cdot))$ as a test function in (6) which leads to the following result.

Corollary 1. *Let $n = 1, \omega \geq 1$. Then for all b satisfying*

$$(7) \quad 0 \geq b > \max_{\alpha > 0} \frac{(2 + \omega^{\frac{q+1}{q-1}})^{1-q} (1 + \alpha^2 \omega)^q - 1 - \alpha^{2q} \omega^{-1}}{2\alpha^q \omega^{-1/2}}$$

the value κ_b^ is attained at a nonnegative fully nontrivial solution of (1). In particular this is true in case*

$$(i) \quad q = 2, b > -\frac{1}{\omega^{3/2} + \sqrt{2(1 + \omega^3)}} \quad \text{or} \quad (ii) \quad \omega = 1, b > \left(\frac{2}{3}\right)^{q-1} - 1.$$

In order to find necessary conditions for the existence of a minimizer one has to estimate the value $b^*(\omega, q)$ from below. For exponents $1 < q \leq 2$ we may combine Theorem 2 (iii) with the following nonexistence result to see that $b^*(\omega, q)$ must be larger than or equal to the right hand side in (8).

Theorem 3. *Let $n = 1, 1 < q \leq 2$ and assume*

$$(8) \quad b < \min_{z > 0} \frac{(\omega^2 - (q-1)\omega)z^{2q} - qz^2 - q\omega^3 z^{2q-2} - (\omega^2(q-1) - \omega)}{qz^{q+2} + (q-2)(\omega^2 + \omega)z^q + q\omega^3 z^{q-2}}.$$

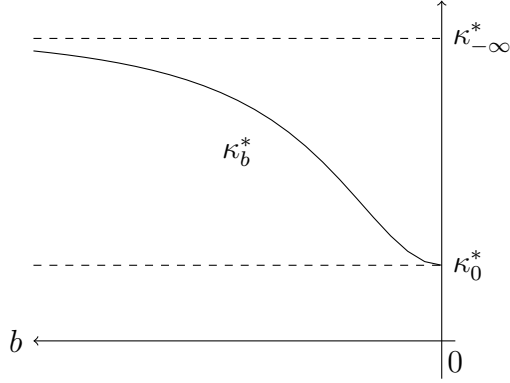
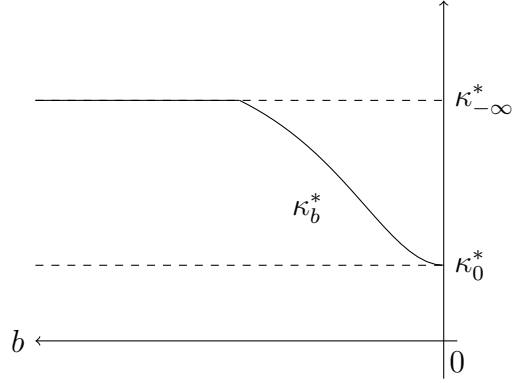
Then the equation (1) does not have any fully nontrivial solution. In particular this holds in case $q = 2, b < -\frac{\omega^2+1}{2\omega}$ or $1 < q \leq 2, \omega = 1, b < -1$.

Remark 1. (i) *It is worth noticing that Theorem 3 not only applies to solutions of minimal energy but to all finite energy solutions.*

(ii) *It would be desirable to know whether a similar nonexistence result is true for exponents larger than 2.*

(iii) *From the strong minimum principle for nonnegative supersolutions of elliptic PDEs we know that the solutions (u, v) of (1) found in Theorem 1 and Theorem 2 satisfy $u > 0$ and $v > 0$ when $q \geq 2$. For $1 < q < 2$ we may apply the minimum principle to the function $u + v$ to conclude that $u + v$ is positive. It seems to be unclear, however, if both u and v are positive functions in that case.*

Finally let us illustrate our main results with two qualitative graphs of the map $b \mapsto \kappa_b^*$ in the cases $n \geq 2$ and $n = 1, 1 < q \leq 2$. The monotonicity of this function is referred to at the end of the first step in the proof of Theorem 1.

(a) energy levels for $n \geq 2$ (b) energy levels for $n = 1, 1 < q \leq 2$

2. NOTATIONS AND CONVENTIONS

In the following we always assume $n \in \mathbb{N}$ and $1 < q < \frac{n}{n-2}$ whenever $n \geq 3$ and $1 < q < \infty$ whenever $n = 1$ or $n = 2$ so that the Sobolev embedding $H_r^1(\mathbb{R}^n) \rightarrow L^{2q}(\mathbb{R}^n)$ exists and is compact in case $n \geq 2$. A function (u, v) is called nontrivial if $u \neq 0$ or $v \neq 0$ and it is called fully nontrivial in case $u \neq 0$ and $v \neq 0$. The same way (u, v) is nonnegative whenever $u \geq 0, v \geq 0$ and it is positive in case $u > 0, v > 0$. We always consider weak radially symmetric solutions of (1) and (3) where it is clear that all solutions of (1) are twice continuously differentiable on \mathbb{R}^n and smooth in the interior of each nodal domain. We use the symbols $\|\cdot\|_r = \|\cdot\|_{L^r(\mathbb{R}^n)}$ to denote the standard Lebesgue norms for $1 \leq r \leq \infty$ and we set $\|(u, v)\| := \sqrt{\|u\|^2 + \|v\|_\omega^2}$ for $u, v \in H_r^1(\mathbb{R}^n)$ where

$$(9) \quad \|u\| := \left(\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 dx \right)^{1/2}, \quad \|v\|_\omega := \left(\int_{\mathbb{R}^n} |\nabla v|^2 + \omega^2 v^2 dx \right)^{1/2}.$$

From the definition of I we get

$$(10) \quad I(u, v) = \frac{q-1}{2q} (\|u\|^2 + \|v\|_\omega^2) \quad \text{for all } (u, v) \in \mathcal{N}_b$$

and in particular for all elements of \mathcal{M}_b or \mathcal{M}_b^* . The function $u_0 \in H_r^1(\mathbb{R}^n)$ is defined as above and for notational convenience we put $c_0 := I(u_0, 0)$. We set $v_0 := \omega^{1/(q-1)} u_0(\omega \cdot)$ so that v_0 is the unique positive solution of $-\Delta v + \omega^2 v = v^{2q-1}$ in \mathbb{R}^n . A short calculation shows

$$I(0, v_0) = \omega^{\frac{2q-n(q-1)}{q-1}} c_0.$$

We will use the facts that the functions u_0, v_0 are minimizers of the functionals $\frac{\|u\|}{\|u\|_{2q}}, \frac{\|v\|_\omega}{\|v\|_{2q}}$, respectively and that all minimizers of these functionals are translates of u_0, v_0 . Moreover, we use that $(u_0, 0)$ is a minimizer of the functional $I|_{\mathcal{N}_b}$ when $b < 0$.

3. PROOF OF THEOREM 1

Throughout this section except for the first step we assume $n \geq 2$ according to the assumptions of Theorem 1. Its proof is given in four steps. First we prove variational characterizations for the values $\kappa_b^*, \kappa_{-\infty}^*$ which turn out to be more convenient than the original ones given by (2) and (4). In the second step we use these characterizations to prove that minimizers of the functionals $I|_{\mathcal{M}_b^*}$ and $I|_{\mathcal{M}_{-\infty}^*}$ exist. In the third step we show that minimizers satisfy the corresponding Euler-Lagrange equation (1) or (3) so that the assertions (i) and (ii) of the theorem are proved. Finally we show part (iii) of the theorem.

Step 1: A more convenient variational characterization for $\kappa_b^, \kappa_{-\infty}^*$ ($n \geq 1$)*

For $s, t > 0$ and $u, v \in H_r^1(\mathbb{R}^n)$ with $u, v \neq 0$ one can check that $(su, tv) \in \mathcal{M}_b^*$ is equivalent to (s, t) being a critical point of the function $\beta_{u,v}$ defined on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and given by

$$\beta_{u,v}(\tilde{s}, \tilde{t}) := I(\tilde{s}u, \tilde{t}v) = \frac{\tilde{s}^2}{2}\|u\|^2 + \frac{\tilde{t}}{2}\|v\|_\omega^2 - \frac{\tilde{s}^{2q}}{2q}\|u\|_{2q}^{2q} - \frac{\tilde{t}^{2q}}{2q}\|v\|_{2q}^{2q} - \frac{b\tilde{s}^q\tilde{t}^q}{q}\|uv\|_q^q.$$

A necessary and sufficient condition for such a critical point to exist is given by

$$(11) \quad \|u\|_{2q}^q \|v\|_{2q}^q + b\|uv\|_q^q > 0.$$

Indeed, in this case the functional $-\beta_{u,v}$ is coercive so that $\beta_{u,v}$ has a global maximum. Moreover one can show that the Hessian of the function $(\tilde{s}, \tilde{t}) \mapsto \beta_{u,v}(\tilde{s}^{1/2q}, \tilde{t}^{1/2q})$ is positive definite on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ so that the maximum is strict and no other critical point can exist. On the other hand a short calculation shows that (11) is also a necessary condition for the existence of a critical point. From

$$\begin{aligned} \max_{s,t>0} \beta_{u,v}(s, t) &= \max_{s,t>0} I(su, tv) \\ &= \max_{\alpha>0} \max_{s>0} I(su, s\alpha v) \\ &= \max_{\alpha>0} \max_{s>0} \frac{s^2}{2}(\|u\|^2 + \alpha^2\|v\|_\omega^2) - \frac{s^{2q}}{2q}(\|u\|_{2q}^{2q} + \alpha^{2q}\|v\|_{2q}^{2q} + 2b\alpha^q\|uv\|_q^q). \\ &= \frac{q-1}{2q} \left(\max_{\alpha>0} \frac{(\|u\|^2 + \alpha^2\|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q}\|v\|_{2q}^{2q} + 2b\alpha^q\|uv\|_q^q} \right)^{\frac{1}{q-1}} \end{aligned}$$

we obtain the following variational characterization for κ_b^* :

$$(12) \quad \kappa_b^* = \inf \left\{ \frac{q-1}{2q} \hat{J}(u, v)^{\frac{1}{q-1}} : u, v \in H_r^1(\mathbb{R}^n), (u, v) \text{ satisfies (11)} \right\}$$

where $\hat{J}(u, v) = \max_{\alpha>0} \frac{(\|u\|^2 + \alpha^2\|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q}\|v\|_{2q}^{2q} + 2b\alpha^q\|uv\|_q^q}.$

Moreover if (u, v) satisfies (11) and minimizes \hat{J} then (su, tv) is a minimizer of $I|_{\mathcal{M}_b^*}$ provided (s, t) is the unique maximizer of $\beta_{u,v}$. Similarly, one can show

$$(13) \quad \begin{aligned} \kappa_{-\infty}^* &= \inf \left\{ \frac{q-1}{2q} \bar{J}(u, v)^{\frac{1}{q-1}} : u, v \in H_r^1(\mathbb{R}^n), u, v \neq 0, uv = 0 \right\} \\ \text{where } \bar{J}(u, v) &= \max_{\alpha > 0} \frac{(\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q} \|v\|_{2q}^{2q}} = \left(\left(\frac{\|u\|}{\|u\|_{2q}} \right)^{\frac{2q}{q-1}} + \left(\frac{\|v\|_\omega}{\|v\|_{2q}} \right)^{\frac{2q}{q-1}} \right)^{q-1}. \end{aligned}$$

Since the constraint $uv = 0$ is more restrictive than (11) we obtain the inequality

$$(14) \quad \kappa_b^* \leq \kappa_{-\infty}^* \quad (b \leq 0).$$

Moreover, from (12) it follows that the map $b \mapsto \kappa_b^*$ is nonincreasing.

Step 2: Existence of nonnegative minimizers

We prove that both κ_b^* and $\kappa_{-\infty}^*$ are attained at nonnegative elements of $\mathcal{M}_b^*, \mathcal{M}_{-\infty}^*$, respectively. By the first step it suffices to show that the functionals \hat{J}, \bar{J} defined in (12), (13) admit fully nontrivial nonnegative minimizers. Since the reasonings for \hat{J} and \bar{J} are almost identical, we only give the proof for \hat{J} .

Let (u_j, v_j) be a minimizing sequence for \hat{J} satisfying (11). Since $\hat{J}(u_j, v_j) = \hat{J}(s|u_j|, t|v_j|)$ for all $s, t > 0$ we may assume $u_j, v_j \geq 0$ as well as $\|u_j\|_{2q} = \|v_j\|_{2q} = 1$. Then (u_j, v_j) is bounded and there is a subsequence (u_j, v_j) that, due to the compactness of the embedding $H_r^1(\mathbb{R}^n) \rightarrow L^{2q}(\mathbb{R}^n)$, converges weakly, almost everywhere and in $L^{2q}(\mathbb{R}^n) \times L^{2q}(\mathbb{R}^n)$ to some nonnegative function (u, v) . This entails $\|u\|_{2q} = \|v\|_{2q} = 1$ as well as $u, v \geq 0$. Furthermore, (u, v) satisfies (11) because otherwise $\hat{J}(u_j, v_j)$ would tend to infinity as $j \rightarrow \infty$ contradicting its property of a minimizing sequence. Hence, for all $\alpha > 0$ we have

$$\begin{aligned} \frac{(\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q} \|v\|_{2q}^{2q} + 2b\alpha^q \|uv\|_q^q} &\leq \liminf_{j \rightarrow \infty} \frac{(\|u_j\|^2 + \alpha^2 \|v_j\|_\omega^2)^q}{\|u_j\|_{2q}^{2q} + \alpha^{2q} \|v_j\|_{2q}^{2q} + 2b\alpha^q \|u_j v_j\|_q^q} \\ &\leq \liminf_{j \rightarrow \infty} \max_{\beta > 0} \frac{(\|u_j\|^2 + \beta^2 \|v_j\|_\omega^2)^q}{\|u_j\|_{2q}^{2q} + \beta^{2q} \|v_j\|_{2q}^{2q} + 2b\beta^q \|u_j v_j\|_q^q}. \end{aligned}$$

Using (12) we find $\hat{J}(u, v) \leq \liminf_{j \rightarrow \infty} \hat{J}(u_j, v_j)$ so that (u, v) is a minimizer of \hat{J} .

Step 3: The solution property

We prove the following two statements:

- (i) In case $b \leq 0$ every minimizer of $I|_{\mathcal{M}_b^*}$ is a solution of (1).
- (ii) Every minimizer of $I|_{\mathcal{M}_{-\infty}^*}$ is a solution of (3).

Let us show assertion (i) first. For $u, v \in H_r^1(\mathbb{R}^n)$ with $u, v \neq 0$ set

$$H_1(u, v) := \|u\|^2 - \|u\|_{2q}^{2q} - b\|uv\|_q^q, \quad \text{and} \quad H_2(u, v) := \|v\|_\omega^2 - \|v\|_{2q}^{2q} - b\|uv\|_q^q.$$

so that $(u, v) \in \mathcal{M}_b^*$ if and only if $H_1(u, v) = H_2(u, v) = 0$. Now, if $(u, v) \in \mathcal{M}_b^*$ is a minimizer of $I|_{\mathcal{M}_b^*}$ then $H_1(u, v) = H_2(u, v) = 0$ implies

$$\begin{aligned} H_1'(u, v)[u, 0] &= 2\|u\|^2 - 2q\|u\|_{2q}^{2q} - qb\|uv\|_q^q = (2 - 2q)\|u\|^2 + bq\|uv\|_q^q < 0, \\ H_2'(u, v)[0, v] &= 2\|v\|_\omega^2 - 2q\|v\|_{2q}^{2q} - qb\|uv\|_q^q = (2 - 2q)\|v\|_\omega^2 + bq\|uv\|_q^q < 0 \end{aligned}$$

so that Lagrange's multiplier rule shows that there are $L_1, L_2 \in \mathbb{R}$ such that

$$(15) \quad I'(u, v) + L_1 H_1'(u, v) + L_2 H_2'(u, v) = 0.$$

It suffices to show $L_1 = L_2 = 0$.

Using $(u, 0), (0, v)$ as test functions in (15) we find $\langle I'(u, v), (u, 0) \rangle = H_1(u, v) = 0$ and $\langle I'(u, v), (0, v) \rangle = H_2(u, v) = 0$ and thus

$$\begin{aligned} 0 &= \left((2 - 2q)\|u\|_{2q}^{2q} + (2 - q)b\|uv\|_q^q \right) L_1 - qb\|uv\|_q^q L_2, \\ 0 &= \left((2 - 2q)\|v\|_{2q}^{2q} + (2 - q)b\|uv\|_q^q \right) L_2 - qb\|uv\|_q^q L_1. \end{aligned}$$

Assume $(L_1, L_2) \neq (0, 0)$. Then $\|uv\|_q^q > 0$ and the determinant of this system vanishes. We therefore get

$$\begin{aligned} 0 &= \left((2 - 2q)\|u\|_{2q}^{2q} + (2 - q)b\|uv\|_q^q \right) \cdot \left((2 - 2q)\|v\|_{2q}^{2q} + (2 - q)b\|uv\|_q^q \right) - (qb\|uv\|_q^q)^2 \\ &= 4(1 - q) \left((b\|uv\|_q^q)^2 - \frac{q - 2}{2} b\|uv\|_q^q (\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q}) - (q - 1)\|u\|_{2q}^{2q}\|v\|_{2q}^{2q} \right). \end{aligned}$$

Solving for $b\|uv\|_q^q < 0$ gives

$$4b\|uv\|_q^q = (q - 2)(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q}) - \sqrt{(q - 2)^2(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q})^2 + 16(q - 1)\|u\|_{2q}^{2q}\|v\|_{2q}^{2q}}.$$

Let now $A, B > 0$ be given by $\|u\|_{2q}^{2q} = A \cdot |b\|uv\|_q^q|$, $\|v\|_{2q}^{2q} = B \cdot |b\|uv\|_q^q|$. Then

$$(16) \quad -4 = (q - 2)(A + B) - \sqrt{(q - 2)^2(A + B)^2 + 16(q - 1)AB}.$$

where A, B are larger than 1 because of

$$\|u\|_{2q}^{2q} - |b\|uv\|_q^q| = \|u\|^2 > 0, \quad \|v\|_{2q}^{2q} - |b\|uv\|_q^q| = \|v\|_\omega^2 > 0.$$

Solving (16) for B we obtain

$$B = \frac{2 + (q - 2)A}{2(q - 1)A - (q - 2)}$$

so that $A > 1$ implies $B < 1$, a contradiction. Hence, the assumption was false, i.e. $I'(u, v) = 0$.

Now consider (ii). Let $(\bar{u}, \bar{v}) \in \mathcal{M}_{-\infty}^*$ be a minimizer of the functional $I|_{\mathcal{M}_{-\infty}^*}$. Due to the one-dimensional Sobolev embedding we may choose \bar{u}, \bar{v} to be a continuous function on $\mathbb{R}^n \setminus \{0\}$ so that the sets $\{\bar{u} \neq 0\}, \{\bar{v} \neq 0\}$ are open. According to the first step we have $\bar{J}(\bar{u}, \bar{v}) \leq \bar{J}(\bar{u} + \varphi, \bar{v} + \psi)$ for all test functions φ, ψ with $\text{supp}(\varphi) \subset \text{supp}(u)$ and $\text{supp}(\psi) \subset \text{supp}(v)$. In view of the second formula for \bar{J} in (13) we find that (\bar{u}, \bar{v}) solves (3).

□

Remark 2. *The above reasoning shows that all critical points and not only minimizers of $I|_{\mathcal{M}_b^*}$ or $I|_{\mathcal{M}_{-\infty}^*}$ satisfy the corresponding Euler-Lagrange equation.*

Step 4: Convergence to a fully nontrivial solution of (3) as $b \rightarrow -\infty$

Let (b_j) be a sequence such that $b_j \rightarrow -\infty$ and let $(u_j, v_j) \in \mathcal{M}_{b_j}^*$ be a sequence of nonnegative fully nontrivial solutions of (1) given by the second step, in particular $I(u_j, v_j) = \kappa_{b_j}^*$. Then (u_j, v_j) is bounded and there is a subsequence (u_j, v_j) that, due to the compactness of the embedding $H_r^1(\mathbb{R}^n) \rightarrow L^{2q}(\mathbb{R}^n)$, converges weakly, almost everywhere and in $L^{2q}(\mathbb{R}^n) \times L^{2q}(\mathbb{R}^n)$ to some nonnegative function (\bar{u}, \bar{v}) . From Sobolev's inequality we get

$$\begin{aligned} \|u_j\|^2 &= \|u_j\|_{2q}^{2q} + b_j \|u_j v_j\|_q^q \leq \|u_j\|_{2q}^{2q} \leq C \|u_j\|^{2q}, \\ \|v_j\|_\omega^2 &= \|v_j\|_{2q}^{2q} + b_j \|u_j v_j\|_q^q \leq \|v_j\|_{2q}^{2q} \leq C \|v_j\|_\omega^{2q} \end{aligned}$$

and thus $\|u_j\|_{2q}, \|v_j\|_{2q} \geq c > 0$ where c, C are positive numbers which do not depend on j . It follows $\|\bar{u}\|_{2q}, \|\bar{v}\|_{2q} \geq c$ and thus $\bar{u}, \bar{v} \neq 0$. In addition we find

$$(17) \quad \|\bar{u}\|^2 \leq \|\bar{u}\|_{2q}^{2q}, \quad \|\bar{v}\|_\omega^2 \leq \|\bar{v}\|_{2q}^{2q}.$$

Since the sequence (u_j, v_j) is bounded we get $\bar{u}\bar{v} \equiv 0$ from

$$\|\bar{u}\bar{v}\|_q^q = \lim_{j \rightarrow \infty} \|u_j v_j\|_q^q = \lim_{j \rightarrow \infty} (\|u_j\|_{2q}^{2q} - \|u_j\|^2) |b_j|^{-1} \leq \liminf_{j \rightarrow \infty} C \cdot |b_j|^{-1} = 0.$$

Furthermore, from (14) we obtain $\kappa_{b_j}^* \leq \kappa_{-\infty}^*$ so that (10) implies

$$\begin{aligned} \frac{q-1}{2q} (\|\bar{u}\|^2 + \|\bar{v}\|_\omega^2) &\leq \frac{q-1}{2q} \liminf_{j \rightarrow \infty} (\|u_j\|^2 + \|v_j\|_\omega^2) \\ &= \liminf_{j \rightarrow \infty} \kappa_{b_j}^* \\ &\leq \limsup_{j \rightarrow \infty} \kappa_{b_j}^* \\ &\leq \kappa_{-\infty}^* \\ &\leq \frac{q-1}{2q} \left(\left(\frac{\|\bar{u}\|}{\|\bar{u}\|_{2q}} \right)^{\frac{2q}{q-1}} + \left(\frac{\|\bar{v}\|_\omega}{\|\bar{v}\|_{2q}} \right)^{\frac{2q}{q-1}} \right) \\ &\leq \frac{q-1}{2q} (\|\bar{u}\|^2 + \|\bar{v}\|_\omega^2) \end{aligned}$$

where we used (13) and (17) in the last two inequalities. Hence, equality occurs in each line and thus $\|\bar{u}\|^2 = \|\bar{u}\|_{2q}^{2q}$, $\|\bar{v}\|_\omega^2 = \|\bar{v}\|_{2q}^{2q}$ as well as $\kappa_{b_j}^* \rightarrow \kappa_{-\infty}^*$, $(u_j, v_j) \rightarrow (\bar{u}, \bar{v})$ as $b \rightarrow -\infty$. This entails $(\bar{u}, \bar{v}) \in \mathcal{M}_{-\infty}^*$ and $I(\bar{u}, \bar{v}) = \kappa_{-\infty}^*$ so that (\bar{u}, \bar{v}) is a minimizer of $I|_{\mathcal{M}_{-\infty}^*}$ and thus a fully nontrivial nonnegative solution of (3). Finally, the assertion follows from

$$\limsup_{j \rightarrow \infty} |b_j| \|u_j v_j\|_q^q = \limsup_{j \rightarrow \infty} \|u_j\|_{2q}^{2q} - \|u_j\|^2 = \|\bar{u}\|_{2q}^{2q} - \|\bar{u}\|^2 = 0.$$

4. PROOF OF THEOREM 2 AND COROLLARY 1

Proof of (i)

First we show $\kappa_{-\infty}^* \geq (2 + \omega^{\frac{q+1}{q-1}})c_0$ and that no element of $\mathcal{M}_{-\infty}^*$ attains this value. Let $(u, v) \in \mathcal{M}_{-\infty}^*$ and in particular $u(0)v(0) = 0$. We first assume $u(0) = 0$. Then the nontrivial functions $u_l := u \cdot 1_{(-\infty, 0)}$, $u_r := u \cdot 1_{(0, \infty)}$ lie in $H^1(\mathbb{R})$, they have disjoint support and satisfy $u_r(r) = u_l(-r)$ due to $u \in H_r^1(\mathbb{R})$. In particular from $\|u\|^2 = \|u\|_{2q}^{2q}$ we infer

$$\|u_l\|^2 = \|u_r\|^2 = \frac{1}{2}\|u\|^2 = \frac{1}{2}\|u\|_{2q}^{2q} = \|u_l\|_{2q}^{2q} = \|u_r\|_{2q}^{2q}.$$

This implies $(u_r, 0), (u_l, 0), (0, v) \in \mathcal{N}_b$ and using (10) as well as $uv \equiv 0$ we obtain

$$\begin{aligned} I(u, v) &= I(u_l, 0) + I(u_r, 0) + I(0, v) \\ &= \frac{q-1}{2q} \cdot (\|u_l\|^2 + \|u_r\|^2 + \|v\|_\omega^2) \\ &= \frac{q-1}{2q} \cdot \left(\left(\frac{\|u_l\|}{\|u_l\|_{2q}} \right)^{\frac{2q}{q-1}} + \left(\frac{\|u_r\|}{\|u_r\|_{2q}} \right)^{\frac{2q}{q-1}} + \left(\frac{\|v\|_\omega}{\|v\|_{2q}} \right)^{\frac{2q}{q-1}} \right). \end{aligned}$$

Since the functions u_0, v_0 minimize the quotients $\frac{\|u\|}{\|u\|_{2q}}, \frac{\|v\|_\omega}{\|v\|_{2q}}$ we get

$$\begin{aligned} I(u, v) &\geq \frac{q-1}{2q} \cdot \left(2 \cdot \left(\frac{\|u_0\|}{\|u_0\|_{2q}} \right)^{\frac{2q}{q-1}} + \left(\frac{\|v_0\|_\omega}{\|v_0\|_{2q}} \right)^{\frac{2q}{q-1}} \right) \\ &= \frac{q-1}{2q} \cdot (2\|u_0\|^2 + \|v_0\|_\omega^2) \\ &= 2I(u_0, 0) + I(0, v_0) \\ &= (2 + \omega^{\frac{q+1}{q-1}})c_0. \end{aligned}$$

Analogously the assumption $v(0) = 0$ leads to

$$I(u, v) \geq (1 + 2\omega^{\frac{q+1}{q-1}})c_0 \geq (2 + \omega^{\frac{q+1}{q-1}})c_0.$$

We therefore get $\kappa_{-\infty}^* \geq (2 + \omega^{\frac{q+1}{q-1}})c_0$. Moreover we find that $\kappa_{-\infty}^*$ is not attained at any element of $\mathcal{M}_{-\infty}^*$ because in case $u(0) = 0$ this would lead to the conclusion that u_r, u_l are translates of u_0 which is impossible because of $\text{supp}(u_r) \cap \text{supp}(u_l) = \emptyset$. A similar reasoning shows that no element (u, v) of $\mathcal{M}_{-\infty}^*$ with $v(0) = 0$ can have energy $(2 + \omega^{\frac{q+1}{q-1}})c_0$.

Now let us prove the opposite inequality. To this end let $\chi_k := \chi(k^{-1} \cdot)$ denote a suitable radially symmetric cut-off function with $\chi \equiv 1$ in $[-1, 1]$ and $\chi \equiv 0$ outside of $(-2, 2)$ then the sequence

$$(u_k, v_k) := \left((u_0 \chi_k)(2k + \cdot) + (u_0 \chi_k)(-2k + \cdot), v_0 \chi_k \right)$$

lies in $\mathcal{M}_{-\infty}^*$ and

$$\lim_{k \rightarrow \infty} I(u_k, v_k) = \lim_{k \rightarrow \infty} (2I(u_0 \chi_k, 0) + I(0, v_0 \chi_k)) = 2I(u_0, 0) + I(0, v_0) = (2 + \omega^{\frac{q+1}{q-1}})c_0$$

which proves $\kappa_{-\infty}^* \leq (2 + \omega^{\frac{q+1}{q-1}})c_0$. Hence, we obtain

$$\kappa_{-\infty}^* = (2 + \omega^{\frac{q+1}{q-1}})c_0.$$

Proof of (ii)

First we prove that $b < b^*(\omega, q)$ implies $\kappa_b^* = \kappa_{-\infty}^*$ and that $0 \geq b > b^*(\omega, q)$ implies $\kappa_b^* < \kappa_{-\infty}^*$. From (i) and the variational characterization for κ_b^* given by (12) we get $\kappa_b^* < \kappa_{-\infty}^*$ if and only if there are functions $u, v \in H_r^1(\mathbb{R}^n)$ with $u, v \neq 0$ and $\|u\|_{2q}^q \|v\|_{2q}^q > |b| \|uv\|_q^q$ that satisfy

$$\max_{\alpha > 0} \frac{(\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q} \|v\|_{2q}^{2q} + 2b\alpha^q \|uv\|_q^q} < \left(\frac{2q}{q-1} \cdot (2 + \omega^{\frac{q+1}{q-1}})c_0 \right)^{q-1} = (2 + \omega^{\frac{q+1}{q-1}})^{q-1} \frac{\|u_0\|_{2q}^{2q}}{\|u_0\|_{2q}^{2q}}.$$

In this case (i) and (13) implies $uv \neq 0$ and thus $b > b^*(\omega, q)$ after some rearrangements of the above inequality. A short argument shows that $0 \geq b > b^*(\omega, q)$ implies $\kappa_b^* < \kappa_{-\infty}^*$.

Let $b < b^*(\omega, q)$ and assume that $\kappa_b^* = \kappa_{-\infty}^*$ is attained at some function (u, v) satisfying the condition (11). Then $uv \neq 0$ since $\kappa_{-\infty}^*$ is not attained, see (i). Choose $\varepsilon > 0$ such that $b + \varepsilon < b^*(\omega, q)$ so that $\kappa_{b+\varepsilon}^* = \kappa_{-\infty}^*$. Then (u, v) satisfies (11) for $b + \varepsilon$ instead of b and we get

$$\begin{aligned} \kappa_b^* &= \frac{q-1}{2q} \left(\max_{\alpha > 0} \frac{(\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q} \|v\|_{2q}^{2q} + 2b\alpha^q \|uv\|_q^q} \right)^{\frac{1}{q-1}} \\ &> \frac{q-1}{2q} \left(\max_{\alpha > 0} \frac{(\|u\|^2 + \alpha^2 \|v\|_\omega^2)^q}{\|u\|_{2q}^{2q} + \alpha^{2q} \|v\|_{2q}^{2q} + 2(b+\varepsilon)\alpha^q \|uv\|_q^q} \right)^{\frac{1}{q-1}} \\ &\geq \kappa_{b+\varepsilon}^* \\ &= \kappa_{-\infty}^* \end{aligned}$$

which contradicts $\kappa_b^* = \kappa_{-\infty}^*$. Hence, κ_b^* is not attained for $b < b^*(\omega, q)$.

Proof of (iii)

In order to prove (iii) we suppose $0 \geq b > b^*(\omega, q)$. From the first statement in the proof of (ii) it follows that this implies

$$(18) \quad \kappa_b^* < \kappa_{-\infty}^* = (2 + \omega^{\frac{q+1}{q-1}})c_0.$$

For these values of b let us investigate the behaviour of a special minimizing sequence for the functional $I|_{\mathcal{M}_b^*}$. We consider the corresponding problem on balls $B_R = (-R, R)$ where R will be sent to infinity. We set $H_{0,r}^1(B_R) := \{u \in H_0^1(B_R) : u \text{ is radially symmetric}\}$. All solutions $(u, v) \in H_{0,r}^1(B_R) \times H_{0,r}^1(B_R)$ of the boundary value problem

$$(19) \quad \begin{aligned} -u'' + u &= |u|^{2q-2}u + b|u|^{q-2}u|v|^q \quad \text{in } B_R, \\ -v'' + \omega^2 v &= |v|^{2q-2}v + b|v|^{q-2}v|u|^q \quad \text{in } B_R, \\ u(-R) &= u(R) = 0, \quad v(-R) = v(R) = 0 \end{aligned}$$

satisfy

$$(20) \quad \int_{B_R} |u'|^2 + u^2 dx = \int_{B_R} |u|^{2q} + b|u|^q|v|^q dx,$$

$$(21) \quad \int_{B_R} |v'|^2 + \omega^2 v^2 dx = \int_{B_R} |v|^{2q} + b|u|^q|v|^q dx.$$

Following the approach of the last section we define

$$\mathcal{M}_b^*(R) := \left\{ (u, v) \in H_{0,r}^1(B_R) \times H_{0,r}^1(B_R) : u, v \neq 0, (u, v) \text{ satisfies (20), (21)} \right\}.$$

As before one can show that $\inf I|_{\mathcal{M}_b^*(R)}$ admits a variational characterization given by

$$(22) \quad \kappa_b^*(R) := \inf_{\mathcal{M}_b^*(R)} I = \inf \left\{ \frac{q-1}{2q} \hat{J}(u, v)^{\frac{1}{q-1}} : u, v \in H_{0,r}^1(B_R), (u, v) \text{ satisfies (11)} \right\}.$$

Using the compactness of the embedding $H_{0,r}^1(B_R) \rightarrow L^{2q}(B_R)$ for all $R > 0$ we obtain the following result:

Proposition 1. *Let $n = 1$. For all $b \leq 0$ the value $\kappa_b^*(R)$ is attained at a fully nontrivial nonnegative solution $(u_R, v_R) \in H_{0,r}^1(B_R) \times H_{0,r}^1(B_R)$ of (19). Moreover, as $R \rightarrow \infty$ we have $\kappa_b^*(R) \rightarrow \kappa_b^*$.*

Proof. The existence of a fully nontrivial nonnegative minimizer of $I|_{\mathcal{M}_b^*(R)}$ can be shown as in the second step in the proof of Theorem 1. From the inclusion $\mathcal{M}_b^*(R) \subset \mathcal{M}_b^*$ it follows

$$(23) \quad \kappa_b^*(R) \geq \kappa_b^*.$$

In order to show $\kappa_b^*(R) \rightarrow \kappa_b^*$ as $R \rightarrow \infty$ we choose a cut-off function χ with $\chi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\chi(x) = 0$ for $|x| \geq 1$, set $\chi_R(x) := \chi(R^{-1}x)$. Then for all $u, v \in H_r^1(\mathbb{R})$ we have $u\chi_R, v\chi_R \in H_{0,r}^1(B_R)$. Moreover, if in addition (u, v) satisfies (11) so does $(u\chi_R, v\chi_R)$ for sufficiently large $R > 0$ and we get from (22) the inequality

$$\limsup_{R \rightarrow \infty} \kappa_b^*(R) \leq \frac{q-1}{2q} \limsup_{R \rightarrow \infty} \hat{J}(u\chi_R, v\chi_R)^{\frac{1}{q-1}} = \frac{q-1}{2q} \hat{J}(u, v)^{\frac{1}{q-1}}.$$

Since this holds for all $u, v \in H_r^1(\mathbb{R})$ satisfying (11) we obtain from (12) the estimate

$$(24) \quad \limsup_{R \rightarrow \infty} \kappa_b^*(R) \leq \kappa_b^*.$$

The inequalities (23) and (24) show $\kappa_b^*(R) \rightarrow \kappa_b^*$ as $R \rightarrow \infty$. \square

Let now $(u_k, v_k) := (u_{R_k}, v_{R_k})$ be the sequence of solutions on $(-R_k, R_k)$ given by Proposition 1 where (R_k) is a fixed positive sequence going off to infinity as $k \rightarrow \infty$. Then (u_k, v_k) lies in $\mathcal{M}_b^*(R_k) \subset \mathcal{M}_b^*$ and we have $I(u_k, v_k) = \kappa_b^*(R_k) \rightarrow \kappa_b^*$ as $k \rightarrow \infty$ by Proposition 1. Since (u_k, v_k) solves (19) on $(-R_k, R_k)$ there is a real number α_k such that

$$(25) \quad -u_k'^2 - v_k'^2 + u_k^2 + \omega^2 v_k^2 - \frac{1}{q}(u_k^{2q} + v_k^{2q} + 2bu_k^q v_k^q) = \alpha_k \quad \text{in } (-R_k, R_k)$$

and $u_k(R_k) = v_k(R_k) = 0$ implies $\alpha_k \leq 0$, see Proposition 2 for the proof of a related result. The sequence (u_k, v_k) is bounded in $H_r^1(\mathbb{R}) \times H_r^1(\mathbb{R})$ and we may choose a subsequence again

denoted by (u_k, v_k) that converges weakly to some $(u, v) \in H_r^1(\mathbb{R}) \times H_r^1(\mathbb{R})$. Then $R_k \rightarrow \infty$ implies that (u, v) is a nonnegative solution of (1) with $I(u, v) \leq \kappa_b^*$. It remains to show $u, v \neq 0$.

We assume $u = 0$. Since u_k, v_k are radially symmetric we have $u'_k(0) = v'_k(0) = 0$. From $b \leq 0$ and (25) we get the inequality

$$0 \geq \alpha_k \geq u_k(0)^2(q - u_k(0)^{2q-2}) + v_k(0)^2(q\omega^2 - v_k(0)^{2q-2}).$$

From $u_k(0) \rightarrow u(0) = 0$ it follows $v_k(0) \geq (q\omega^2)^{\frac{1}{2q-2}}$ for almost all k and hence $v(0) > 0$. It follows that $(u, v) = (0, v)$ is a solution of (1) satisfying $v(0) > 0$ as well as $v \geq 0$. Kwong's uniqueness result [8] gives $v = v_0$ and we obtain

$$(26) \quad (u_k, v_k) \rightharpoonup (0, v_0), \quad (u_k, v_k) \rightarrow (0, v_0) \text{ in } C_{loc}^2(\mathbb{R}).$$

Let now $x_k \in [0, R_k]$ be given by

$$\max_{[-R_k, R_k]} u_k = u_k(x_k) = u_k(-x_k) > 0.$$

From the differential equation (19) and $b \leq 0$ we infer

$$0 \leq -\frac{u_k''(x_k)}{u_k(x_k)} = \frac{u_k(x_k)^{2q-1} + bu_k(x_k)^{q-1}v_k(x_k)^q - u_k(x_k)}{u_k(x_k)} \leq u_k(x_k)^{2q-2} - 1$$

and thus

$$(27) \quad u_k(x_k) = u_k(-x_k) \geq 1.$$

From (26), (27) we get $x_k \rightarrow +\infty$. Let now $(\tilde{u}_k^+, \tilde{v}_k^+), (\tilde{u}_k^-, \tilde{v}_k^-)$ be given by

$$(\tilde{u}_k^+, \tilde{v}_k^+) := (u_k(\cdot + x_k), v_k(\cdot + x_k)), \quad (\tilde{u}_k^-, \tilde{v}_k^-) := (u_k(\cdot - x_k), v_k(\cdot - x_k)).$$

These sequences are bounded in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and there are subsequences again denoted by $(\tilde{u}_k^+, \tilde{v}_k^+), (\tilde{u}_k^-, \tilde{v}_k^-)$ that converge weakly and locally uniformly to nonnegative functions $(\tilde{u}^+, \tilde{v}^+), (\tilde{u}^-, \tilde{v}^-)$, respectively. The inequality (27) implies $\tilde{u}^+(0), \tilde{u}^-(0) > 0$. Since the functions $(\tilde{u}^+, \tilde{v}^+), (\tilde{u}^-, \tilde{v}^-)$ are nontrivial solutions of (1) on $(-\infty, a), (-a, \infty)$, respectively where $a := \lim_{k \rightarrow \infty} (R_k - x_k)$ we obtain $(\tilde{u}^\pm, \tilde{v}^\pm) \in \mathcal{N}_b$ and (10) gives

$$(28) \quad \frac{q-1}{2q}(\|\tilde{u}^\pm\|^2 + \|\tilde{v}^\pm\|_\omega^2) = I(\tilde{u}^\pm, \tilde{v}^\pm) \geq \min_{\mathcal{N}_b} I = I(u_0, 0) = c_0.$$

Now let χ denote a cut-off-function satisfying $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$, set $\chi_R(x) := \chi(R^{-1}x)$. Choosing $k_0(R)$ sufficiently large we obtain $x_k > 2R$ for all $k \geq k_0(R)$. In particular for all $k \geq k_0(R)$ the sets $\text{supp}(\chi_R)$, $\text{supp}(\chi_R(\cdot - x_k))$, $\text{supp}(\chi_R(\cdot + x_k))$ are pairwise disjoint and we get

$$\begin{aligned} & \| (u_k, v_k) \|^2 \\ & \geq \| (u_k \chi_R, v_k \chi_R) \|^2 + \| (u_k \chi_R(\cdot - x_k), v_k \chi_R(\cdot - x_k)) \|^2 + \| (u_k \chi_R(\cdot + x_k), v_k \chi_R(\cdot + x_k)) \|^2 \\ & = \| (u_k \chi_R, v_k \chi_R) \|^2 + \| (\tilde{u}_k^+ \chi_R, \tilde{v}_k^+ \chi_R) \|^2 + \| (\tilde{u}_k^- \chi_R, \tilde{v}_k^- \chi_R) \|^2. \end{aligned}$$

From $(u_k, v_k) \rightharpoonup (0, v_0)$, $(\tilde{u}_k^+, \tilde{v}_k^+) \rightharpoonup (\tilde{u}^+, \tilde{v}^+)$ and $(\tilde{u}_k^+, \tilde{v}_k^+) \rightharpoonup (\tilde{u}^-, \tilde{v}^-)$ we infer

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|(u_k, v_k)\|^2 &\geq \|(0, v_0 \chi_R)\|^2 + \|(\tilde{u}^+ \chi_R, \tilde{v}^+ \chi_R)\|^2 + \|(\tilde{u}^- \chi_R, \tilde{v}^- \chi_R)\|^2 \\ &= \|v_0 \chi_R\|_\omega^2 + \|(\tilde{u}^+ \chi_R, \tilde{v}^+ \chi_R)\|^2 + \|(\tilde{u}^- \chi_R, \tilde{v}^- \chi_R)\|^2. \end{aligned}$$

Since this inequality holds for all $R > 0$ we obtain

$$\liminf_{k \rightarrow \infty} \|(u_k, v_k)\|^2 \geq \|v_0\|_\omega^2 + \|(\tilde{u}^+, \tilde{v}^+)\|^2 + \|(\tilde{u}^-, \tilde{v}^-)\|^2$$

and from the estimate (28) and (10) we get

$$\begin{aligned} \kappa_b^* &= \lim_{k \rightarrow \infty} \kappa_b^*(R_k) \\ &= \frac{q-1}{2q} \lim_{k \rightarrow \infty} \|(u_k, v_k)\|^2 \\ &\geq \frac{q-1}{2q} \left(\|v_0\|_\omega^2 + \|(\tilde{u}^+, \tilde{v}^+)\|^2 + \|(\tilde{u}^-, \tilde{v}^-)\|^2 \right) \\ &\geq (2 + \omega^{\frac{q+1}{q-1}}) c_0 \end{aligned}$$

which contradicts (18). Hence, $u \neq 0$. Analogously the assumption $v = 0$ leads to the inequality

$$\kappa_b^* \geq (1 + 2\omega^{\frac{q+1}{q-1}}) c_0 \geq (2 + \omega^{\frac{q+1}{q-1}}) c_0,$$

which again gives a contradiction. It follows $u, v \neq 0$ and the proof is finished. \square

Proof of Corollary 1

Assume that b is larger than the right hand side in (7). According to Theorem 2 (iii) it suffices to show that this implies $b > b^*(\omega, q)$. To this end we estimate $b^*(\omega, q)$ from above using the test function $(u, v) := (u_0, u_0(\omega \cdot))$ in (6). We obtain

$$\begin{aligned} b^*(\omega, q) &\leq \max_{\alpha > 0} \frac{(2 + \omega^{\frac{q+1}{q-1}})^{1-q} \|u_0\|^{-2q} \|u_0\|_{2q}^{2q} (\|u_0\|^2 + \alpha^2 \|u_0(\omega \cdot)\|_\omega^2)^q - \|u_0\|_{2q}^{2q} - \alpha^{2q} \|u_0(\omega \cdot)\|_{2q}^{2q}}{2\alpha^q \|u_0 u_0(\omega \cdot)\|_q^q} \\ &= \max_{\alpha > 0} \frac{(2 + \omega^{\frac{q+1}{q-1}})^{1-q} (1 + \alpha^2 \omega)^q - 1 - \alpha^{2q} \omega^{-1}}{2\alpha^q} \cdot \frac{\|u_0\|_{2q}^{2q}}{\|u_0 u_0(\omega \cdot)\|_q^q}. \end{aligned}$$

The numerator function is bounded from above by its negative maximum $2^{1-q} - 1$ which is attained at $\alpha = 2^{-1/2} \omega^{\frac{1}{q-1}}$. In particular, the right hand side is negative for all $\alpha > 0$ so that the estimate $\|u_0 u_0(\omega \cdot)\|_q^q \leq \|u_0\|_{2q}^q \|u_0(\omega \cdot)\|_{2q}^q = \omega^{-1/2} \|u_0\|_{2q}^{2q}$ leads to

$$b^*(\omega, q) \leq \max_{\alpha > 0} \frac{(2 + \omega^{\frac{q+1}{q-1}})^{1-q} (1 + \alpha^2 \omega)^q - 1 - \alpha^{2q} \omega^{-1}}{2\alpha^q \omega^{-1/2}}$$

where the right hand side is smaller than b by the assumption of Corollary 1. As indicated above the result now follows from Theorem 2 (iii).

Finally, in the special cases $q = 2$ or $1 < q \leq 2, \omega = 1$ we may determine the value of the right hand side in (7) explicitly. In case $q = 2$ the maximum is attained at $\alpha = \left(\frac{(1+\omega^3)\omega}{2}\right)^{1/4}$ and we get

$$b^*(\omega, 2) \leq -\frac{1}{\omega^{3/2} + \sqrt{2(1+\omega^3)}}.$$

In case $1 < q \leq 2, \omega = 1$ the maximum is attained at $\alpha = 1$ and we obtain the value

$$b^*(1, q) \leq \left(\frac{2}{3}\right)^{q-1} - 1.$$

5. PROOF OF THEOREM 3

In the proof of Theorem 3 we will need the following elementary result.

Proposition 2. *Let $n = 1, \omega \geq 1$. Then every solution $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of (1) satisfies*

$$(29) \quad -u'^2 - v'^2 + u^2 + \omega^2 v^2 - \frac{1}{q}(|u|^{2q} + |v|^{2q} + 2b|u|^q|v|^q) = 0 \quad \text{in } \mathbb{R}.$$

Proof. For a solution (u, v) of (1) the derivative of the left hand side in (29) exists and equals zero. Hence there is some $\alpha \in \mathbb{R}$ such that

$$-u'^2 - v'^2 + u^2 + \omega^2 v^2 - \frac{1}{q}(|u|^{2q} + |v|^{2q} + 2b|u|^q|v|^q) = \alpha \quad \text{in } \mathbb{R}.$$

If α were not equal to zero then there would exist $\delta > 0$ such that $u'^2 + v'^2 + u^2 + \omega^2 v^2 \geq \delta$ in \mathbb{R} which contradicts $u, v \in H^1(\mathbb{R})$. \square

Proof of Theorem 3

Let $b \in \mathbb{R}$ satisfy the inequality (8). We assume that there is a fully nontrivial solution $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of (1). Since the functions $(-u, v), (u, -v), (-u, -v)$ solve (1), too, we may assume that a maximal open interval $A \subset \{x \in \mathbb{R} : u(x) > 0, v(x) > 0\}$ is non-empty. We will prove later that the assumptions of the theorem imply that every critical point of $u^\omega v$ in A is strict local minimizer. Once this is shown a contradiction can be achieved in the following way.

In case $u^\omega v$ does not have any critical point in A the function $u^\omega v$ is monotone on A so that A is unbounded and $(u^\omega v)(x)$ does not converge to 0 as $|x| \rightarrow \infty$. This contradicts $u, v \in H^1(\mathbb{R})$. If, however, a critical point $x_0 \in A$ exists, then x_0 is a strict local minimizer and therefore it must be the only critical point because any other critical point would have to be a strict local minimizer, too. It follows that $u^\omega v$ is increasing on $(x_0, \infty) \cap A$ and decreasing on $(-\infty, x_0) \cap A$ so that $(u^\omega v)(x) \geq (u^\omega v)(x_0) > 0$ for all $x \in A$. Hence, $A = \mathbb{R}$ from the maximality of A and thus $(u^\omega v)(x) \geq (u^\omega v)(x_0) > 0$ for all $x \in \mathbb{R}$ which contradicts $u, v \in H^1(\mathbb{R})$.

Now we show that every critical point of $u^\omega v$ in A is a strict local minimizer. Clearly, for $x \in A$ such that $(u^\omega v)'(x) = 0$ we have

$$(30) \quad \omega u'(x)v(x) = -u(x)v'(x)$$

and a short calculation gives

$$(31) \quad u'(x)v'(x) = -\omega u(x)v(x) \cdot \frac{u'(x)^2 + v'(x)^2}{u(x)^2 + \omega^2 v(x)^2}.$$

Using (30) we obtain at the point x

$$\begin{aligned} (u^\omega v)'' &= \omega(\omega - 1)u^{\omega-2}u'^2v + \omega u''u^{\omega-1}v + 2\omega u'u^{\omega-1}v' + u^\omega v'' \\ &= u^{\omega-1} \left(\omega \frac{u'}{u} \cdot (\omega - 1)u'v + \omega u''v + 2\omega u'v' + uv'' \right) \\ &= u^{\omega-1} \left(-(\omega - 1)u'v' + \omega u''v + 2\omega u'v' + uv'' \right) \\ &= u^{\omega-1} \left((\omega + 1)u'v' + \omega u''v + uv'' \right). \end{aligned}$$

From (31) and the partial differential equation satisfied by (u, v) we get

$$\begin{aligned} (u^\omega v)'' &= u^{\omega-1} \left(-\omega(\omega + 1)uv \cdot \frac{u'^2 + v'^2}{u^2 + \omega^2 v^2} \right. \\ &\quad \left. + \omega uv(1 - u^{2q-2} - bu^{q-2}v^q) + uv(\omega^2 - v^{2q-2} - bv^{q-2}u^q) \right). \end{aligned}$$

Proposition 2 gives

$$\begin{aligned} (u^\omega v)'' &= u^\omega v \left(-\omega(\omega + 1) \cdot \left(1 - \frac{u^{2q} + v^{2q} + 2bu^q v^q}{q(u^2 + \omega^2 v^2)} \right) \right. \\ &\quad \left. + \omega - \omega u^{2q-2} - b\omega u^{q-2}v^q + \omega^2 - v^{2q-2} - bv^{q-2}u^q \right) \\ &= \frac{u^\omega v}{q(u^2 + \omega^2 v^2)} \cdot \left(\omega(\omega + 1)(u^{2q} + v^{2q} + 2bu^q v^q) - q(u^2 + \omega^2 v^2) \cdot \right. \\ &\quad \left. (\omega u^{2q-2} + b\omega u^{q-2}v^q + v^{2q-2} + bv^{q-2}u^q) \right) \\ &= \frac{u^\omega v}{q(u^2 + \omega^2 v^2)} \cdot \left(-bqu^{q+2}v^{q-2} + (\omega^2 - (q-1)\omega)u^{2q} - qu^2v^{2q-2} \right. \\ &\quad \left. + b(2-q)(\omega^2 + \omega)u^q v^q - q\omega^3 u^{2q-2}v^2 \right. \\ &\quad \left. - (\omega^2(q-1) - \omega)v^{2q} - bq\omega^3 u^{q-2}v^{q+2} \right) \\ &= \frac{u^\omega v^{2q+1}}{q(u^2 + \omega^2 v^2)} \cdot \left(-bqz^{q+2} + (\omega^2 - (q-1)\omega)z^{2q} - qz^2 \right. \\ &\quad \left. + b(2-q)(\omega^2 + \omega)z^q - q\omega^3 z^{2q-2} \right. \\ &\quad \left. - (\omega^2(q-1) - \omega) - bq\omega^3 z^{q-2} \right) \end{aligned}$$

where $z := \frac{u(x)}{v(x)}$. From assumption (8) and $u(x), v(x), z > 0$ we obtain $(u^\omega v)''(x) > 0$ which proves the claim. We finish the proof of Theorem 3 considering the special cases $q = 2$ and $1 < q < 2, \omega = 1$.

In case $q = 2$ the minimum in (8) is attained at $z = \sqrt{\omega}$ and we obtain that fully nontrivial solutions do not exist for parameter values $b < -\frac{\omega^2+1}{2\omega}$. In case $1 < q < 2, \omega = 1$ we find that the following inequality holds for $b \leq -1$ and all $z > 0$

$$\begin{aligned} & (-b) \cdot (qz^{q+2} - 2(2-q)z^q + qz^{q-2}) + (2-q)z^{2q} - qz^2 - qz^{2q-2} - (q-2) \\ & \geq 1 \cdot (qz^{q+2} - 2(2-q)z^q + qz^{q-2}) + (2-q)z^{2q} - qz^2 - qz^{2q-2} - (q-2) \\ & = (2-q)(z^q - 1)^2 + q(z^2 - z^{q-2})(z^q - 1) \\ & \geq 0 \end{aligned}$$

with equality if and only if $b = -1$ and $z = 1$. Rearranging terms we see that the minimum in (8) is 1 and it is attained at $z = 1$. We obtain the nonexistence result for $b < -1$. \square

Remark 3. *In the above reasoning we did not use the assumption $1 < q \leq 2$ explicitly. Nevertheless we had to exclude the case $q > 2$ case because the minimum in (8) does not exist. Indeed, sending z to 0 and using $\omega \geq 1 > \frac{1}{q-1}$ we find that the infimum is $-\infty$.*

REFERENCES

- [1] A. Ambrosetti and E. Colorado. Bound and ground states of coupled nonlinear Schrödinger equations. *C. R. Math. Acad. Sci. Paris*, 342(7):453–458, 2006.
- [2] T. Bartsch, N. Dancer, and Z.-Q. Wang. A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. *Calc. Var. Partial Differential Equations*, 37(3-4):345–361, 2010.
- [3] M. Conti and V. Felli. Minimal coexistence configurations for multispecies systems. *Nonlinear Anal.*, 71(7-8):3163–3175, 2009.
- [4] M. Conti and V. Felli. Global minimizers of coexistence for competing species. *J. Lond. Math. Soc. (2)*, 83(3):606–618, 2011.
- [5] M. Conti, S. Terracini, and G. Verzini. A variational problem for the spatial segregation of reaction-diffusion systems. *Indiana Univ. Math. J.*, 54(3):779–815, 2005.
- [6] D. G. de Figueiredo and O. Lopes. Solitary waves for some nonlinear Schrödinger systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1):149–161, 2008.
- [7] T. Dohnal and H. Uecker. Coupled mode equations and gap solitons for the 2D Gross-Pitaevskii equation with a non-separable periodic potential. *Physica D Nonlinear Phenomena*, 238:860–879, 2009.
- [8] M. K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbf{R}^n . *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.
- [9] T.-C. Lin and J. Wei. Ground state of N coupled nonlinear Schrödinger equations in \mathbf{R}^n , $n \leq 3$. *Comm. Math. Phys.*, 255(3):629–653, 2005.
- [10] T.-C. Lin and J. Wei. Erratum: “Ground state of N coupled nonlinear Schrödinger equations in \mathbf{R}^n , $n \leq 3$ ” [Comm. Math. Phys. 255 (2005), no. 3, 629–653; mr2135447]. *Comm. Math. Phys.*, 277(2):573–576, 2008.
- [11] L. A. Maia, E. Montefusco, and B. Pellacci. Positive solutions for a weakly coupled nonlinear Schrödinger system. *J. Differential Equations*, 229(2):743–767, 2006.
- [12] B. Sirakov. Least energy solitary waves for a system of nonlinear Schrödinger equations in \mathbf{R}^n . *Comm. Math. Phys.*, 271(1):199–221, 2007.

- [13] H. Tavares and S. Terracini. Sign-changing solutions of competition-diffusion elliptic systems and optimal partition problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(2):279–300, 2012.
- [14] J. Wei and T. Weth. Nonradial symmetric bound states for a system of coupled Schrödinger equations. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 18(3):279–293, 2007.
- [15] J. Wei and T. Weth. Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Ration. Mech. Anal.*, 190(1):83–106, 2008.

R. MANDEL

DEPARTMENT OF MATHEMATICS, KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)

D-76128 KARLSRUHE, GERMANY

E-mail address: `Rainer.Mandel@kit.edu`